

Random walks and random fixed-point free involutions

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2001 J. Phys. A: Math. Gen. 34 L381

(<http://iopscience.iop.org/0305-4470/34/28/101>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.97

The article was downloaded on 02/06/2010 at 09:08

Please note that [terms and conditions apply](#).

LETTER TO THE EDITOR

Random walks and random fixed-point free involutions

T H Baker^{1,3} and P J Forrester²

¹ Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

² Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia

E-mail: p.forrester@ms.unimelb.edu.au

Received 5 June 2001

Published 6 July 2001

Online at stacks.iop.org/JPhysA/34/L381

Abstract

A bijection is given between fixed-point free involutions of $\{1, 2, \dots, 2N\}$ with maximum decreasing subsequence size $2p$ and two classes of vicious (non-intersecting) random walker configurations confined to the half-line lattice points $l \geq 1$. In one class of walker configurations the maximum displacement of the rightmost walker is p . Because the scaled distribution of the maximum decreasing subsequence size is known to be in the soft edge GOE (random real symmetric matrices) universality class, the same holds true for the scaled distribution of the maximum displacement of the rightmost walker.

PACS numbers: 0540F, 0210A, 0550

Random permutations are fundamental combinatorial objects, which are intimately related to other fundamental combinatorial objects such as Young tableaux via the Robinson–Schensted–Knuth correspondence. We recall that a Young tableau can be regarded as a numbered diagram of a partition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \geq 0$. The diagram consists of squares drawn within a matrix array with a square drawn in each row ($1 \leq j \leq p$) and column k ($1 \leq k \leq \lambda_j$), while in each square is recorded a number specified by some rule. Recently random permutations, Young tableaux and their generalizations have been shown to be at the core of certain statistical mechanical models of growth processes [14–16, 19], vicious walker paths [5, 10, 13, 17] and exclusion processes (the latter via mappings to certain growth processes and vicious walker paths) amongst other topics. This has led to progress in the study of these statistical mechanical models, by way of the progress in the determination of fluctuation formulae for quantities associated with random permutations [2–4].

³ Present address: CMIS, 108 North Rd, Australian National University, Canberra 2601, Australia.

As an example of the insight gained, we draw attention to the work of Prähofer and Spohn [19]. These authors identify distinct scaling forms for growth models in the Kardar–Parisi–Zhang (KPZ) universality class, that is growth models described by the KPZ equation

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x} \right)^2 + \xi(t)$$

where $\xi(t)$ is a noise term. If the growth profile is curved, the fluctuations are conjectured to coincide with the distribution of the largest eigenvalue in the scaled GUE (random Hermitian matrices), while if the growth profile has zero curvature the fluctuations are conjectured to coincide with the distribution of the largest eigenvalue in the scaled GOE (random real symmetric matrices). A matrix X from the GOE ($\beta = 1$) or GUE ($\beta = 2$) is specified by elements chosen with a joint distribution proportional to the Gaussian $\exp(-\beta X^2/2)$. The largest eigenvalue occurs in the neighbourhood of $\lambda = \sqrt{2N}$ (which is referred to as the soft edge), and by making the scaling $\lambda \mapsto \sqrt{2N} + \lambda/\sqrt{2N}^{1/6}$ the corresponding correlation functions have well defined limits [9]. Moreover, in the scaled $N \rightarrow \infty$ limit of both the GOE and GUE the distribution of the largest eigenvalue is known exactly in terms of a certain Painlevé II transcendent [23, 24]. This identification was formulated from an exact correspondence between a particular model of KPZ growth—the polynuclear growth model—and increasing subsequences of random permutations. The latter have been proved to have GUE soft edge fluctuations [1, 6, 15, 18] in the absence of further constraints, but GOE soft edge fluctuations in the presence of the symmetry constraint restricting the permutations to fixed-point free involutions. The two cases correspond to a curved and zero-curvature interface respectively in the corresponding polynuclear growth model.

In this letter we identify a statistical mechanical model for which the profile displacement can be put into correspondence with the maximum decreasing subsequence length of fixed-point free involutions. As this quantity has been shown rigorously to have GOE soft edge fluctuations [3], it follows that the profile of the statistical mechanical model is in the GOE soft edge universality class. We remark that the polynuclear growth model from a flat substrate is also in correspondence with the maximum decreasing subsequence length of fixed-point free involutions [19], the mapping being quite direct (unlike the present case). The model to be considered is the random-turns model of vicious random walkers. This model can be viewed either as a two-dimensional lattice model of non-intersecting directed paths, or as a stochastic model of hard-core particles on a lattice in one dimension. In the latter picture, at discrete time intervals $t = 1, 2, \dots$ a particle which has a vacant site as its left neighbour or its right neighbour (or both neighbours) is selected at random and moved to the vacant neighbouring site (if both sites are vacant, either is chosen with equal probability). Plotting the trajectories of the particles on an l - t diagram (l labelling the lattice sites) gives the directed, non-intersecting paths picture of the model.

Our interest is in two classes of configurations of this walker model. The first is when there are exactly p walkers, initially equally spaced on neighbouring lattice sites $l = 1, 2, \dots, p$ and furthermore constrained to the region $l \geq 1$ (in the vicious walker vernacular, at the site $l = 0$ there is a cliff at which the walkers fall to their death [7]). It is required that after $2N$ steps the walkers return to their initial sites.

In the second class of configurations the walkers again begin on the neighbouring sites $l = 1, 2, \dots$, are confined to the region $l \geq 1$, and return to their initial sites after $2N$ steps. But rather than there being p walkers there are now N^* walkers with $N^* \geq N$. The parameter p enters by the requirement that the rightmost walker has a maximum displacement of no more than p lattice sites from its initial position. On the other hand, the value of N^* is not a relevant parameter because only a maximum of N consecutive walkers, counted from the rightmost

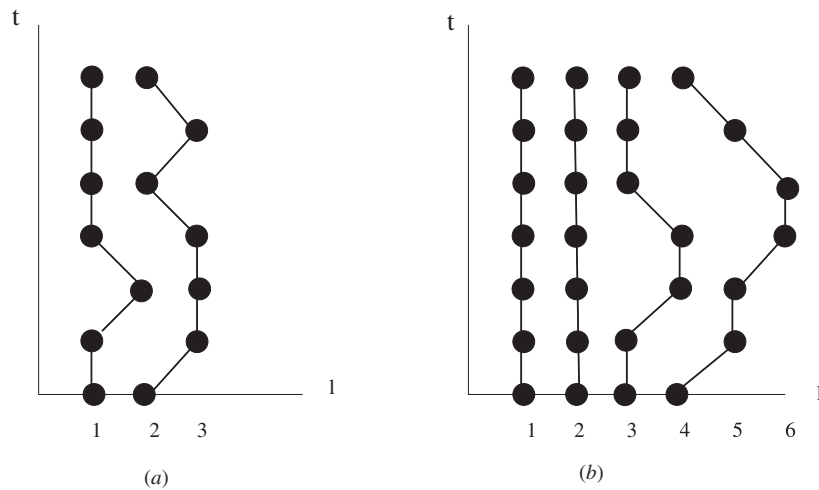


Figure 1. (a) There are $p = 2$ walkers which move a total of $2N = 6$ steps. (b) There are $N^* = 4$ walkers which move a total of $2N = 6$ steps with the maximum displacement of the rightmost walker given by $p = 2$.

walker, move from their initial sites. An example of the first and second class of configurations is given in figure 1.

Our first result is that both these classes of configurations are in one-to-one correspondence with a certain subclass of fixed-point free involutions.

Proposition 1. *For both of the two classes of vicious walker configurations specified above there is a bijection with fixed-point free involutions of $\{1, 2, \dots, 2N\}$ (i.e. permutations consisting solely of two cycles) constrained so that the length of the maximum decreasing subsequence is less than or equal to $2p$.*

We will give the details of the bijection for the second class of configurations, and afterwards indicate the modification required to establish the bijection for the first class of configurations. Denote the walker initially at lattice site j by $N^* + 1 - j$ ($j = 1, 2, \dots, N^*$). A walker configuration can then be coded as a sequence of integers from the alphabets $1, 2, \dots, N^*$ and $\bar{1}, \bar{2}, \dots, \bar{N}^*$, with the occurrence of the integer j (\bar{j}) at position t denoting that walker j (\bar{j}) moved one site to the right (left) at time step t .

Of course, not all words of length $2N$ from this alphabet give rise to legal walker configurations. For a legal configuration, at each time step we must have that

$$n_1 \geq n_2 \geq \dots \geq n_{N^*} \geq 0 \tag{1}$$

where

$$n_j := \#j's - \#\bar{j}'s$$

and after time step $2N$, each n_j must equal zero. Requirement (1) can be represented diagrammatically as the diagram of a (conjugate) partition in which column j is of length n_j (see figure 2). A successive sequence of diagrams so generated (starting and finishing with the empty diagram \emptyset) uniquely specifies the walk, thus demonstrating a bijection between such diagrams and walk configurations.

In the theory of Young tableaux, the diagrams so generated are examples of oscillating tableaux. In general, these tableaux map onto certain two-line arrays [22], which in the present

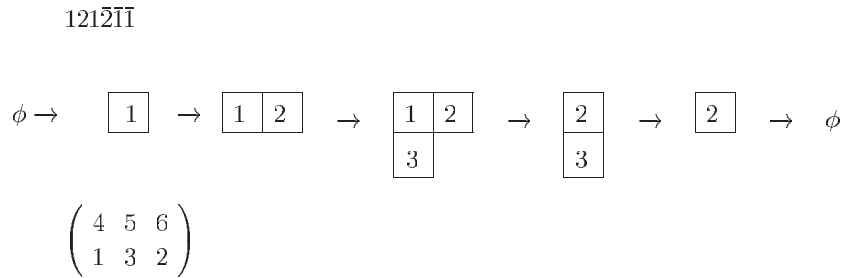


Figure 2. The word corresponding to the walker configuration (b) of figure 1, the sequence of oscillating tableaux corresponding to the word, and the two-line array constructed from the oscillating tableaux.

case represent involutions of $\{1, 2, \dots, 2N\}$ with no fixed points. To construct the array, we number the box i if it is added to the diagram at step i . If, instead, a box is removed at step i (say from column j) this is to be done via the procedure of reverse column insertion, which means that if the particular box ejected, x_i say, was then inserted by the Schensted column insertion procedure the original diagram would be restored (see [12] for a description of the Schensted algorithm). The fact that the removal occurred at step i is recorded by putting the pair (i, x_i) into a two-line array with i on top. Note that since x_i was bumped out at step i , it must have been inserted in an earlier step, so $x_i < i$. Furthermore, all numbers in the array will be distinct, and at the end of the procedure there will be N pairs from the numbers $\{1, 2, \dots, 2N\}$ with the top numbers ordered $i_1 < i_2 < \dots < i_N$. This procedure is illustrated in figure 2. The pairs forming the array can be considered as the two cycles in a fixed-point free involution of $\{1, 2, \dots, 2N\}$.

The constraint that the rightmost walker have maximum displacement of exactly p lattice sites to the right of its starting position means that the maximum length of the first column of each tableau is less than or equal to p boxes. Because each tableau has the number of the boxes strictly increasing down each column and across each row, it follows from the reverse column bumping procedure used to form the corresponding two-line array that the maximum decreasing subsequence length in the bottom line is precisely the maximum size of the first column (see the example of figure 2) which is less than or equal to p . Hence, the walker configurations are in one-to-one correspondence with the two-line arrays already noted, subject to the additional constraint that the maximum decreasing subsequence length in the bottom line is less than or equal to p . In the correspondence between the two-line array and fixed-point free involutions, this constrains the fixed-point free involutions to have maximum decreasing subsequence length less than or equal to $2p$. To see this we note that the fixed-point free involution can be constructed by extending the top line of the two-line array to include all integers $1, 2, \dots, 2N$ in order and filling in the bottom line according to pairings implied by the original two-line array. We see that if

$$x_{j_1} > x_{j_2} > \dots > x_{j_q}$$

is a particular decreasing subsequence of maximum length q ($q \leq p$) in the bottom line of the original two-line array, then the increasing subsequence of length $2q$ formed from

$$\{j_1, j_2, \dots, j_q\} \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_q}\}$$

in the top line of the new two-line array gives a decreasing subsequence of length $2q$ in the bottom line of the new two-line array. This construction worked in reverse shows that no decreasing subsequence in the fixed-point free involution can have length greater than $2q$.

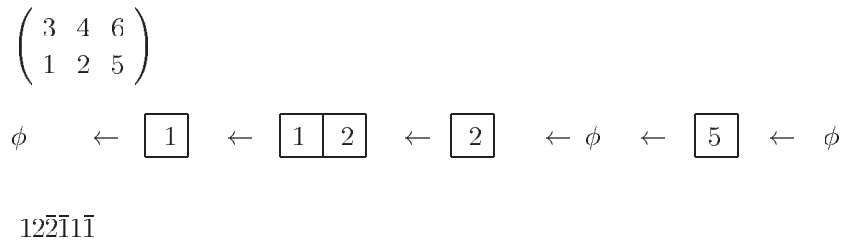


Figure 3. Correspondence between a two-line array corresponding to a fixed-point free involution and a sequence of oscillating tableaux, and the correspondence between the oscillating tableaux and a word. The word is equivalent to walker configuration (a) of figure 1, but translated at least one lattice site to the right so that $N^* > N$, with stationary walkers filling the intervening sites to the left down to $l = 1$.

The above procedure associating each walker configuration with a two-line array is reversible in that, starting with a two-line array of the type specified, a unique sequence of oscillating tableaux, and thus walker configuration, can be constructed. Following [22] we work backwards in the construction of the two-line array from the sequence of oscillating tableaux. In going from the tableau at step i to that at step $i - 1$ there are two distinct situations. One is that i does not appear in the top row of the two-line array, indicating that the tableau at step i was not the result of removing a box from the tableau at step $i - 1$, but rather came from adding a box labelled i to the tableau at step $i - 1$. Thus, deleting the box labelled i from the tableau at step i gives the tableau at step $i - 1$. On the other hand, we may have that i does appear in the top row of the array, being part of the pair (i, x_i) . In this case the tableau at step i was obtained from the tableau at step $i - 1$ as a result of an inverse column bumping which ejected x_i . Thus, the tableau at step $i - 1$ is constructed from the tableau at step i by Schensted column inserting x_i . An example of this inverse procedure is given in figure 3. From the rules of the column insertion, the maximum attained length of the first column of the tableaux will equal the length of the largest decreasing subsequence in the bottom line of the two-line array and thus be less than or equal to p .

Hence, for every sequence of oscillating tableaux, starting and finishing with the empty tableau and having column length less than or equal to p , there is a two-line array equivalent to a fixed-point free involution having maximum decreasing subsequence length less than or equal to $2p$, and furthermore the correspondence can be established in the reverse direction. Because there is a bijection between the oscillating tableaux and random walker configurations, the result of proposition 1 for the second class of walker configurations is established.

Let us now turn our attention to the first class of configurations. The walker configurations are again written as words, this time from the alphabets $1, 2, \dots, p$ and $\bar{1}, \bar{2}, \dots, \bar{p}$. The constraint (1) (with N^* replaced by p) is represented as a diagram but now with row j of length n_j rather than column j as previously, this feature being the essential difference between the two cases. Note that the length of the first column now represents the number of walkers displaced from their initial conditions.

As before, the boxes are numbered by i if added at time step i , and removed via the reverse column bumping procedure, with the fact that the removal occurred at step i recorded by putting the pair (i, x_i) into a two-line array with i on top. An example is given in figure 4. The constraint that there be less than or equal to p walkers restricts the first column length to be less than or equal to p . As already noted, the fact that the reverse column bumping procedure is used to construct the two-line array from the tableau implies the former must therefore have decreasing subsequence length no greater than p . Thus, each walker configuration can be mapped to a

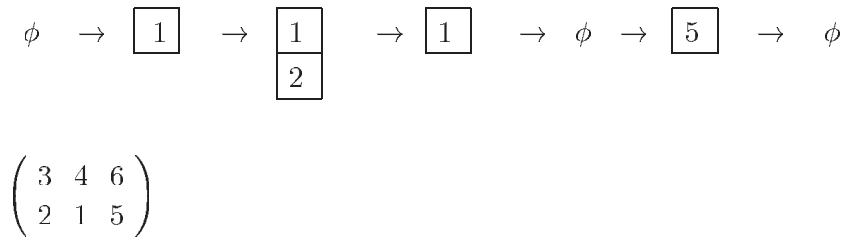


Figure 4. The oscillating tableaux for the configuration (a) of figure 1, or equivalently the word of figure 3, constructed from the rules for a fixed p of walkers, and the corresponding two-line array.

unique two-line array of the same type as occurred in the corresponding mapping for the second class of configurations. Furthermore, we have detailed how to associate such two-line arrays with a unique sequence of oscillating tableaux. From this sequence of oscillating tableaux we can construct the word corresponding to the walker configuration. Note that this differs from the construction in the case of the second class of configurations because now it is row j which specifies the moves of walker j . The final result is that there is a bijection between the first class of walker configurations and two-line arrays with top line ordered $i_1 < i_2 < \dots < i_N$, and maximum decreasing subsequence length no greater than p . The latter being equivalent to fixed-point free involutions of $\{1, 2, \dots, 2N\}$ with maximum decreasing subsequence length no greater than $2p$, we see that proposition 1 is now established.

The second class of configurations count the number of walker configurations with a specific bound p on the maximum displacement of the rightmost walker. From the l - t diagrams of figure 1 we see that this is equivalent to counting the number of configurations which give rise to a growth profile with a bound p on its maximum spread, L_N say. Our interest is in the distribution of L_N . Now, with L_t denoting the displacement of the rightmost walker after t steps, the symmetry of the configurations under $t \rightarrow 2N - t$ means the statistical properties of L_t are the same as those of L_{2N-t} . At the centre of symmetry will be the maximum displacement L_N , and the conjecture of Prähofer and Spohn predicts GUE fluctuations if the profile is curved at this point, or GOE fluctuations if the profile has zero curvature. Unfortunately, the analytic form of the profile is not known, so we cannot make use of this prediction presently.

In fact, the nature of the fluctuations can be determined rigorously by using the bijection of proposition 1 between the second class of configurations and fixed-point free involutions. In particular, we can make use of the known distribution of the maximum decreasing subsequence length for fixed-point free involutions of $\{1, 2, \dots, 2N\}$ to deduce the limiting distribution of L_N . Regarding the former, let L_N^{inv} denote the maximum decreasing subsequence length, and define the scaled quantity

$$\chi_N^{\text{inv}} := \frac{L_N^{\text{inv}} - 2\sqrt{2N}}{(\sqrt{2N})^{1/6}} = \frac{L_N^{\text{inv}}/2 - 2\sqrt{N/2}}{(\sqrt{N/2})^{1/6}}.$$

Then it is proved in [3] that

$$\lim_{N \rightarrow \infty} \Pr(\chi_N^{\text{inv}} \leq x) = F_1(x)$$

where $F_1(x)$ denotes the cumulative distribution of the largest eigenvalue of matrices from the scaled GOE [24] (see [11] for a simplified derivation). The following result is then an immediate consequence of proposition 1.

Proposition 2. *Let L_N denote the maximum displacement of the rightmost walker in the second*

class of random walker configurations specified above, and set

$$\chi_N := \frac{L_N - \sqrt{2N}}{\frac{1}{2}(2N)^{1/6}}.$$

Then

$$\lim_{N \rightarrow \infty} \Pr(\chi_N \leq x) = F_1(x).$$

Hence, the walker profile at its maximum width exhibits GOE fluctuations. The converse of the prediction of Prähofer and Spohn would then imply that the walker profile has zero curvature at this point.

As a final issue we consider the p -dimensional integral formula for the number, $f_{Np}^{(inv)}$ say, of fixed-point free involutions of $\{1, 2, \dots, 2N\}$ constrained so that the length of the maximum decreasing subsequence is less than or equal to $2p$. With $USp(p)$ denoting the group of $2p \times 2p$ unitary symplectic matrices (or equivalently the group of $p \times p$ unitary matrices with real quaternion elements), it was shown by Rains [21] that

$$\begin{aligned} f_{Np}^{(inv)} &= \langle \text{Tr}(S)^{2N} \rangle_{S \in USp(p)} \\ &= \frac{1}{(2\pi)^p p!} \int_0^\pi d\theta_1 \cdots \int_0^\pi d\theta_p \left(\sum_{j=1}^p 2 \cos \theta_j \right)^{2N} \prod_{j=1}^p |1 - z_j^2|^2 \\ &\quad \times \prod_{1 \leq j < k \leq p} |1 - z_j z_k|^2 |z_j - z_k|^2 \end{aligned} \tag{2}$$

where $z_j = e^{i\theta_j}$.

Formula (2) is in fact a special case of a counting formula for a class of vicious walker paths. Thus, consider p vicious walkers in the lock step model, confined to the lattice sites $l \geq 1$, starting at positions

$$1 \leq l_1^{(0)} < l_2^{(0)} < \cdots < l_p^{(0)} \tag{3}$$

and arriving at positions

$$1 \leq l_1 < l_2 < \cdots < l_p \tag{4}$$

after n steps. With $Z_n(l_1^{(0)}, \dots, l_p^{(0)}; l_1, \dots, l_p)$ denoting the number of distinct walker configurations of this prescription, we have the following result.

Proposition 3.

$$\begin{aligned} Z_n(l_1^{(0)}, \dots, l_p^{(0)}; l_1, \dots, l_p) &= \frac{1}{(2\pi)^p} \int_{-\pi}^\pi d\theta_1 \cdots \int_{-\pi}^\pi d\theta_p \left(\sum_{j=1}^p 2 \cos \theta_j \right)^n \\ &\quad \times \det [e^{i(l_j - l_k^{(0)})\theta_j} - e^{i(l_j + l_k^{(0)})\theta_j}]_{j,k=1, \dots, p}. \end{aligned} \tag{5}$$

Analogous to the proof of a similar counting formula in [8], this can be verified by first noting from the definition of the particular lock step model that

$$Z_n(l_1^{(0)}, \dots, l_p^{(0)}; l_1, \dots, l_p) := Z_n(l_1, \dots, l_p)$$

is the unique solution of the multi-dimensional difference equation

$$\begin{aligned} Z_{n+1}(l_1, \dots, l_p) &= Z_n(l_1 - 1, l_2, \dots, l_p) + Z_n(l_1, l_2 - 1, \dots, l_p) \\ &\quad + \cdots + Z_n(l_1, l_2, \dots, l_p - 1) \\ &\quad + Z_n(l_1 + 1, l_2, \dots, l_p) + Z_n(l_1, l_2 + 1, \dots, l_p) \\ &\quad + \cdots + Z_n(l_1, l_2, \dots, l_p + 1) \end{aligned} \tag{6}$$

subject to the non-intersection condition

$$Z_n(l_1, \dots, l_p) = 0 \quad \text{if } l_j = l_k \quad (j \neq k) \tag{7}$$

the constraint $l_j \geq 1$ ($j = 1, \dots, p$) which requires

$$Z_n(l_1, \dots, l_p) = 0 \quad \text{if } l_1 = 0 \tag{8}$$

(here use has been made of the ordering (4)) and the initial condition

$$Z_n(l_1^{(0)}, \dots, l_p^{(0)}; l_1, \dots, l_p) = \prod_{k=1}^p \delta_{l_k^{(0)}, l_k} \tag{9}$$

where again use has been made of the orderings (3) and (4).

To verify that (5) satisfies (6) we note that (5) gives

$$\begin{aligned} Z_{n+1}(l_1, \dots, l_p) &= \left(\frac{1}{2\pi}\right)^p \sum_{\mu=1}^p \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_p (e^{i\theta_\mu} + e^{-i\theta_\mu}) \\ &\times \left(\sum_{j=1}^p 2 \cos \theta_j\right)^n \det [e^{i(l_j - l_k^{(0)})\theta_j} - e^{i(l_j + l_k^{(0)})\theta_j}]_{j,k=1,\dots,p}. \end{aligned} \tag{10}$$

Using the fact that

$$\begin{aligned} e^{\pm i\theta_\mu} \det [e^{i(l_j - l_k^{(0)})\theta_j} - e^{i(l_j + l_k^{(0)})\theta_j}]_{j,k=1,\dots,p} \\ = \det \begin{bmatrix} e^{i(l_{j_1} - l_{k_1}^{(0)})\theta_{j_1}} - e^{i(l_{j_1} + l_{k_1}^{(0)})\theta_{j_1}} \\ e^{i(l_{\mu \pm 1} - l_{k_1}^{(0)})\theta_{\mu \pm 1}} - e^{i(l_{\mu \pm 1} + l_{k_1}^{(0)})\theta_{\mu \pm 1}} \\ e^{i(l_{j_2} - l_{k_1}^{(0)})\theta_{j_2}} - e^{i(l_{j_2} + l_{k_1}^{(0)})\theta_{j_2}} \end{bmatrix}_{\substack{j_1=1,\dots,\mu-1 \\ j_2=\mu+1,\dots,p \\ k=1,\dots,p}} \end{aligned}$$

and recalling (5) we can immediately identify the right-hand side of (10) with the right-hand side of (5).

To verify (7) we simply note that if $l_j = l_k$ for any $j \neq k$ then two rows of the matrix in (5) are the same so the determinant vanishes. Condition (9) is a property of (5) since with $l_1 = 0$ the integrand is odd in θ_1 and thus the integral vanishes. Finally, to verify the initial condition (9) we make use of the definition of a determinant

$$\det[a_{jk}]_{j,k=1,\dots,p} = \sum_{P \in S_p} \varepsilon(P) \prod_{j=1}^p a_{jP(j)}$$

where $\varepsilon(P)$ denotes the parity of the permutation P , to expand the integrand in (5) and integrate term by term. Recalling that each $l_j^{(0)}$ and l_k is positive, this gives

$$Z_0(l_1^{(0)}, \dots, l_p^{(0)}; l_1, \dots, l_p) = \sum_{P \in S_p} \varepsilon(P) \prod_{j=1}^p \delta_{l_j, l_{P(j)}}. \tag{11}$$

The ordering constraints (3) and (4) imply that all terms in (11) except for the identity permutation must vanish, and so (7) is indeed satisfied.

Since the difference equation, the boundary conditions and the initial conditions are all satisfied by (5), we conclude that (5) correctly represents $Z_n(l_1, \dots, l_p)$.

Although Z_n is a positive integer, the integrand in (5) is complex. An integral representation with a positive real integrand can, in the case $l_j^{(0)} = l_j$ ($j = 1, \dots, p$), be obtained by first noting

$$\begin{aligned} Z_{n_1+n_2}(l_1^{(0)}, \dots, l_p^{(0)}; l_1^{(0)}, \dots, l_p^{(0)}) \\ = \sum_{1 \leq l_1 < l_2 < \dots < l_p} Z_{n_1}(l_1^{(0)}, \dots, l_p^{(0)}; l_1, \dots, l_p) Z_{n_2}(l_1, \dots, l_p; l_1^{(0)}, \dots, l_p^{(0)}). \end{aligned} \tag{12}$$

From (5), and after simple manipulation of the determinant therein, we see that

$$\begin{aligned}
 & Z_{n_1}(l_1^{(0)}, \dots, l_p^{(0)}; l_1, \dots, l_p) Z_{n_2}(l_1, \dots, l_p; l_1^{(0)}, \dots, l_p^{(0)}) \\
 &= \frac{1}{(2\pi)^{2p}} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_p \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_p \left(\sum_{j=1}^p 2 \cos \theta_j \right)^{n_1} \\
 &\quad \times \left(\sum_{j=1}^p 2 \cos \phi_j \right)^{n_2} \prod_{j=1}^p e^{il_j(\theta_j - \phi_j)} \det [e^{-il_k^{(0)}\theta_j} - e^{il_k^{(0)}\theta_j}]_{j,k=1,\dots,p} \\
 &\quad \times \det [e^{il_k^{(0)}\phi_j} - e^{-il_k^{(0)}\phi_j}]_{j,k=1,\dots,p}. \tag{13}
 \end{aligned}$$

This is an even symmetric function of the l_j , which vanish for $l_j = l_k$ and $l_j = 0$. Consequently, the sum in (12) can be replaced by

$$\frac{1}{2^p p!} \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \cdots \sum_{l_p=-\infty}^{\infty}.$$

Performing the sum in (13) using

$$\sum_{l=-\infty}^{\infty} e^{il(\theta_j - \phi_j)} = 2\pi \delta(\theta_j - \phi_j) \quad |\theta_j - \phi_j| < 2\pi$$

and putting $n_1 + n_2 = n$ gives

$$\begin{aligned}
 & Z_n(l_1^{(0)}, \dots, l_p^{(0)}; l_1^{(0)}, \dots, l_p^{(0)}) \\
 &= \frac{1}{(2\pi)^p} \frac{1}{2^p p!} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_p \left(\sum_{j=1}^p 2 \cos \theta_j \right)^n \\
 &\quad \times \left| \det [e^{il_k^{(0)}\theta_j} - e^{-il_k^{(0)}\theta_j}]_{j,k=1,\dots,p} \right|^2. \tag{14}
 \end{aligned}$$

We can use (14) to rederive (2) since from the definitions we have

$$f_{Np}^{inv} = Z_{2N}(l_1^{(0)}, \dots, l_p^{(0)}; l_1, \dots, l_p) \Big|_{l_j^{(0)}=l_j=j(j=1,\dots,p)}. \tag{15}$$

Setting $l_j^{(0)} = j$ ($j = 1, \dots, p$) in (14), noting that with $z_j = e^{i\theta_j}$ we have from the type C Vandermonde formula [20]

$$\left| \det [z_j^k - z_j^{-k}]_{j,k=1,\dots,p} \right|^2 = \prod_{j=1}^p |1 - z_j^2|^2 \prod_{1 \leq j < k \leq p} |1 - z_j z_k|^2 |z_j - z_k|^2 \tag{16}$$

and making use of the fact that the integrand is even, we see that formula (2) indeed results.

PJF acknowledges the financial support of the Australian Research Council, and that of funds obtained by Professor K Aomoto for his visit to Japan in June 2000 which facilitated the present collaboration.

References

[1] Baik J, Dieft P and Johansson K 1999 On the distribution of the length of the longest increasing subsequence of random permutations *J. Am. Math. Soc.* **12** 1119–78
 [2] Baik J and Rains E M 1999 Algebraic aspects of increasing subsequences *Preprint* math.CO/9905083
 [3] Baik J and Rains E M 1999 The asymptotics of monotone subsequences of involutions *Preprint* math.CO/9905084
 [4] Baik J and Rains E M 1999 Symmetrized random permutations *Preprint* math.CO/9910019
 [5] Baik J 2000 Random vicious walks and random matrices *Commun. Pure Appl. Math.* **53** 1385–410

- [6] Borodin A, Okounkov A and Olshanski G 1999 Asymptotics of Plancherel measures for symmetric groups *J. Am. Math. Soc.* **13** 481–515
- [7] Forrester P J 1989 Probability of survival for vicious walkers near a cliff *J. Phys. A: Math. Gen.* **22** L609–13
- [8] Forrester P J 1991 Exact results for vicious walker models of domain walls *J. Phys. A: Math. Gen.* **24** 203–18
- [9] Forrester P J 1993 The spectrum edge of random matrix ensembles *Nucl. Phys. B* **402** 709–28
- [10] Forrester P J 1999 Random walks and random permutations *Preprint* math.CO/9907037
- [11] Forrester P J 2000 Painlevé transcendent evaluation of the scaled distribution of the smallest eigenvalue in the Laguerre orthogonal and symplectic ensembles *Preprint* nlin.SI/0005064
- [12] Fulton W 1997 *Young Tableaux (London Mathematical Society Student Texts)* (Cambridge: Cambridge University Press)
- [13] Guttmann A J, Owczarek A L and Viennot X G 1998 Vicious walkers and Young tableaux I: without walls *J. Phys. A: Math. Gen.* **31** 8123–35
- [14] Tracy C A, Gravner J and Widom H 2000 Limit theorems for height fluctuations in a class of discrete space and time fluctuations *Preprint* math.PR/0005133
- [15] Johansson K 1999 Discrete orthogonal polynomial ensembles and the Plancherel measure *Preprint* math.CO/9906120
- [16] Johansson K 2000 Shape fluctuations and random matrices *Commun. Math. Phys.* **209** 437–76
- [17] Krattenthaler C, Guttmann A J and Viennot X G 2000 Vicious walkers, friendly walkers and Young tableaux: II *J. Phys. A: Math. Gen.* **33** 8835–66
- [18] Okounkov A 2000 Random matrices and random permutations *Int. Math. Res. Notices* **20** 1043–95
- [19] Prähofer M and Spohn H 2000 Universal distributions for growth processes in 1 + 1 dimensions and random matrices *Phys. Rev. Lett.* **84** 4882–5
- [20] Proctor R A 1988 Odd symplectic groups *Inv. Math.* **92** 307–32
- [21] Rains E M 1998 Increasing subsequences and the classical groups *Elect. J. Combin.* **5** #R12
- [22] Sundaram S 1990 The Cauchy identity for $Sp(2n)$ *J. Combin. Theory A* **53** 209–38
- [23] Tracy C A and Widom H 1994 Level-spacing distributions and the Airy kernel *Commun. Math. Phys.* **159** 151–74
- [24] Tracy C A and Widom H 1996 On orthogonal and symplectic matrix ensembles *Commun. Math. Phys.* **177** 727–54